ELSEVIER Stochastic Processes and their Applications 69 (1997) 275–287

stochastic processes and their applications

The power laws of M and N in greedy lattice animals

Sungchul Lee

Department of Mathemaics, National University of Singapore, Singapore 119260, Singapore

Received 7 November 1996; received in revised form 12 March 1997

Abstract

We consider the power laws of certain limiting values in greedy lattice animals which were introduced by Cox, Gandolfi, Griffin, and Kesten (1993) and Gandolfi and Kesten (1994). We study the behavior of the limiting values as we change the parameter p. © 1997 Elsevier Science B.V.

AMS classification: primary 60G50; secondary 60K35

Keywords: Lattice animals; Self-avoiding paths

1. Introduction

Let Z^d be a d-dimensional cubic lattice. The *distance* between $u \in Z^d$ and $v \in Z^d$ is defined by

$$\|u - v\| = \|u - v\|_1 = \sum_{i=1}^{d} |u(i) - v(i)|.$$

Define

$$B(u,l) = \left\{ v \in Z^d : \sup_{1 \leqslant i \leqslant d} |u(i) - v(i)| \leqslant l \right\},$$

and

$$\partial B(u,l) = \left\{ v \in Z^d : \sup_{1 \leqslant i \leqslant d} |u(i) - v(i)| = l \right\}.$$

In the case $u = 0$, we simplify the notations $B(0,l)$ and $\partial B(0,l)$ by $B(l)$ and $\partial B(l)$, respectively.

A sequence $\pi = (v_1, \ldots, v_n)$ in Z^d is a *path* if $\|v_{i+1} - v_i\| = 1$ for $1 \leqslant i \leqslant n-1$. The *length* of the path $\pi = (v_1, \ldots, v_n)$ is n and is denoted by $|\pi|$. Note that the length $|\pi|$ of a path π is not defined in a usual way since we count not the edges but the vertices in the path. If a path $\pi = (v_1, \ldots, v_n)$ satisfies $v_i \neq v_j$ for $i \neq j$, it is called *self-avoiding*.

A subset η of Z^d is a *lattice animal* (or *connected*) if for any $x, y \in \eta$ there is a path $\pi = (v_1, \ldots, v_n)$ in η from $v_1 = x$ to $v_n = y$. The *size* of the lattice animal η is the cardinality of the subset η of Z^d and is denoted by $|\eta|$.

0304-4149/97/$17.00
PII S0304-4149(97)00047-1

Let $\{X_v(p) : v \in Z^d\}$ be i.i.d. Bernoulli random variables with a success parameter p, i.e.,

$$X_v(p) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p. \end{cases}$$

We take $d \geqslant 2$ to avoid trivialities. Consider the random subset C of Z^d which is obtained by deleting all vertices v with $X_v(p) = 0$. For two disjoint subsets $\mathscr{A}$ and $\mathscr{B}$ of Z^d, we denote the event that some vertex $u_1 \in \mathscr{A}$ is connected to some vertex $u_2 \in \mathscr{B}$ by a path $\pi = (v_1, \ldots, v_n)$ in C by

$$\mathscr{A} \leftrightarrow \mathscr{B}.$$

We also denote the event $\bigcap_{n=1}^{\infty} \{0 \leftrightarrow \partial B(n)\}$ that for any $n \geqslant 1$ the origin 0 is connected to some vertex $v_n \in \partial B(n)$ by a path $\pi = (v_1, \ldots, v_n)$ in C by

$$0 \leftrightarrow \infty.$$

The fundamental theorem of percolation is that there exists $0 < p_c = p_c(d) < 1$ such that

$$P_p(0 \leftrightarrow \infty) = \begin{cases} 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases} \tag{1.1}$$

p_c is called the *critical probability* for site percolation on Z^d (see Grimmett, 1989, Chapter 1 for more details).

If $p < p_c$, then the connected component C_0 of C containing the origin consists of only finite number of vertices and so

$$\lim_{n\to\infty} P_p(0 \leftrightarrow \partial B(n)) = 0.$$

The behavior of this convergence is one of the main interests in percolation: There exist strictly positive finite constants ρ and σ, which are independent of p, and $\xi(p)$, which depends on p, such that for $0 < p < p_c$

$$\rho n^{1-d} \mathrm{e}^{-n/\xi(p)} \leqslant P_p(0 \leftrightarrow \partial B(n)) \leqslant \sigma n^{d-1} \mathrm{e}^{-n/\xi(p)}, \tag{1.2}$$

and such that

$$\lim_{p \uparrow p_c} \xi(p) = \infty. \tag{1.3}$$

Moreover,

$$\lim_{n\to\infty} \frac{\log P_p(0 \leftrightarrow (n, 0, \ldots, 0))}{n} = -\frac{1}{\xi(p)}. \tag{1.4}$$

$\xi(p)$ is called the *correlation length* (see Grimmett, 1989, Chapter 5 for more details).

Cox et al. (1993) introduce

$$M_n(p) = \max\left\{\sum_{v\in\pi} X_v(p) : \pi \in \Pi(n)\right\},$$

$$N_n(p) = \max\left\{\sum_{v\in\eta} X_v(p) : \eta \in E(n)\right\}, \tag{1.5}$$

where $\Pi(n)$ is the set of selfavoiding paths π of length n starting at the origin and $E(n)$ is the set of lattice animals η of size n containing the origin. Gandolfi and Kesten (1994) show that there exist positive finite constants $M(p)$ and $N(p)$ such that

$$\lim_{n\to\infty} \frac{M_n(p)}{n} = M(p) \quad \text{and} \quad \lim_{n\to\infty} \frac{N_n(p)}{n} = N(p) \text{ a.s. and in } L^1. \tag{1.6}$$

For the motivation and the development of the greedy lattice animal model, see Cox et al. (1993) and Gandolfi and Kesten (1994). For the applications of the model, see Fontes and Newman (1993) and Sznitman (1995).

By the definition of $M(p)$ and $N(p)$, $M(0)=N(0)=0$,

$$p \leqslant M(p) \leqslant N(p), \quad 0<p<1,$$

$M(1)=N(1)=1$. Gandolfi and Kesten (1994) point out that

$$p<M(p)\leqslant N(p), \quad 0<p<1,$$

and Lee (1993) shows that

$$p<M(p)<N(p), \quad 0<p<p_c,$$

and that, for $p_c \leqslant p \leqslant 1$, $M(p)=N(p)=1$. However, we have very little information on the dependence of $M(p)$ and $N(p)$ on p. To understand the dependence of $M(p)$ and $N(p)$ on p, Lee (1996) studies the continuity property of $M(p)$ and $N(p)$. By an obvious coupling, we may restate the results of Lee (1993) and Lee (1996) as follows (one can find the proof in Lee, 1994).

Theorem 1. *$M(\cdot)$ and $N(\cdot)$ are increasing continuous functions of p such that $M(0)=N(0)=0$,*

$$p<M(p)<N(p)<1, \quad 0<p<p_c,$$

$M(p)=N(p)=1$, $p_c \leqslant p \leqslant 1$. Moreover, these two functions are strictly increasing for $p \leqslant p_c$.

Here, we study the behaviors of $M(p)$ and $N(p)$ as $p\downarrow 0$ and as $p \uparrow p_c$. The direct motivation of this paper comes from Fontes and Newman (1993). Fontes and Newman (1993, Corollary to Theorem 3), show that for any $\varepsilon>0$ there exists a strictly positive finite constant C_ε such that

$$M(p) \leqslant N(p) \leqslant C_\varepsilon p^{1/(d+\varepsilon)}.$$

This raises a question whether $M(p)$ and $N(p)$ behave like a power of p as $p\downarrow 0$. We have the answer to this problem.

Theorem 2. *There exist strictly positive finite constants $C_1 = C_1(d)$ and $C_2 = C_2(d)$ such that*

$$C_1 p^{1/d} \leqslant M(p) \leqslant N(p) \leqslant C_2 p^{1/d}. \tag{1.7}$$

We also get some partial results on the behavior of $M(p)$ and $N(p)$ for $p \uparrow p_c$.

Theorem 3. *Fix $0 < p_0 < p_c$. Then, there exists a strictly positive finite constant $C_3 = C_3(p_0, d)$ such that*

$$\frac{p}{p_c} \leqslant M(p) < N(p) \leqslant 1 - \frac{C_3}{(\xi(p)\log\xi(p))^d}, \quad p_0 < p < p_c. \tag{1.8}$$

If $d = 2$, then there exists a strictly positive finite constant $C_4 = C_4(p_0, d)$ such that

$$1 - \frac{C_4}{\xi(p)} \leqslant M(p) < N(p) \leqslant 1 - \frac{C_3}{(\xi(p)\log\xi(p))^2}, \quad p_0 < p < p_c. \tag{1.9}$$

In Section 2, we prove Theorem 2. We construct an infinite (random) path in a certain optimal way and evaluating the values $X_v(p)$ along this infinite path we obtain a lower bound for $M(p)$. We also count systematically the number of lattice animals of interest and we use the Borel–Cantelli lemma to obtain an upper bound for $N(p)$. In Section 3, we prove Theorem 3. By a coupling we get a lower bound for $M(p)$. If $d = 2$, by a well-known result from first-passage percolation on the square lattice Z^2 we get a better lower bound for $M(p)$. For an upper bound for $N(p)$ we employ Peierls argument.

2. Power laws of $M(p)$ and $N(p)$ as $p\downarrow 0$

In this section, we prove Theorem 2. This consists of two parts: First, we construct an infinite (random) path in a certain optimal way and evaluating the values $X_v(p)$ along this infinite path we obtain a lower bound for $M(p)$. Second, we count systematically the number of lattice animals of interest and we use the Borel–Cantelli lemma to obtain an upper bound for $N(p)$.

We begin with a lower bound for $M(p)$. We construct an infinite self-avoiding (random) path $\pi = (v_1, v_2, \ldots)$ in Z^d which starts at the origin and which moves "as directly as possible" from one vertex v with $X_v(p) = 1$ to another vertex v' with $X_{v'}(p) = 1$: Let $u_1^l = (-1, 0, \ldots, 0)$ and let $u_1 = (0, 0, \ldots, 0)$. Once u_n^l and u_n have been determined, we order the points of $u_n + (Z^+)^d = \{v \in Z^d : v(i) \geqslant u_n(i), 1 \leqslant i \leqslant d\}$ in such a way that

$$v \succ_n v' \quad \text{if} \ \sum_{i=1}^{d} v(i) > \sum_{i=1}^{d} v'(i). \tag{2.1}$$

Then, we let u_{n+1}^l be the first vertex v in $u_n + (Z^+)^d$ in the given order such that $X_v(p) = 1$ and let $u_{n+1} = u_{n+1}^l + (1, 0, \ldots, 0)$. We construct a selfavoiding (random) path π_n from u_n to u_{n+1}^l in which we first move $u_{n+1}^l(1)-u_n(1)$ steps in the direction of the positive first coordinate axis, then $u_{n+1}^l(2)-u_n(2)$ steps in the direction of the positive second coordinate axis, and so on. Now, we construct an infinite selfavoiding (random) path π, which starts at the origin, by concatenating the π_n's. Then, by (1.5) and (1.6)

$$M(p) \geqslant \limsup_{n\to\infty} \frac{n}{\sum_{m=1}^{n}|\pi_m|}. \tag{2.2}$$

The right-hand side of (2.2) will be estimated by the strong law of large numbers, Jensen's inequality, and a coupling.

Let $\{Y_n(p) : n \geqslant 1\}$ be i.i.d. Bernoulli random variables with a success parameter p and let $\{l_n(p) : n \geqslant 1\}$ be defined inductively by

$$l_1(p) = \inf\{k \geqslant 1 : Y_k(p) = 1\} \tag{2.3}$$

and for $n \geqslant 2$

$$l_n(p) = \inf\{k \geqslant 1 : Y_{l_{n-1}(p)+k}(p) = 1\}. \tag{2.4}$$

Then, $\{l_n(p): n \geqslant 1\}$ are i.i.d. strictly positive integer-valued random variables with mean p^{-1}. Then, by the strong law of large numbers

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} l_m(p)}{n} = p^{-1} \quad \text{a.s.,}$$

and so by the strong law of large numbers and by Jensen's inequality

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} l_m^{1/d}(p)}{n} \leqslant p^{-1/d} \quad \text{a.s.} \tag{2.5}$$

Now, let us do a coupling. When we choose the u_n^l's and u_n's, we test $X_v(p)$ in a certain order: We test $X_v(p)$, $v \in u_1 + (Z^+)^d$, in the given order until we find the first vertex u_2^l such that $X_{u_2^l}(p) = 1$. Next, we test $X_v(p)$, $v \in u_2 + (Z^+)^d$, in the given order until we find the first vertex u_3^l such that $X_{u_3^l}(p) = 1$, and so on. We define $Y_n(p)$ as the nth $X_v(p)$ in this testing order. Note that the vertex at which the nth testing is done is random except the case $n = 1$ in which $Y_1(p) = X_0(p)$ and that we never have to test $X_v(p)$ at the same vertex twice. Note also that $\{Y_n(p) : n \geqslant 1\}$ are i.i.d. Bernoulli random variables with a success parameter p. Now, we define $\{l_n(p) : n \geqslant 1\}$ by (2.3) and (2.4) as above. If we test, after choosing u_n^l, $l_n(p)$ values of $X_v(p)$ in the given order to choose u_{n+1}^l, then the length $|\pi_n|$ of the rectilinear path π_n from u_n to u_{n+1}^l is less than or equal to $C(d) l_n^{1/d}(p)$. Indeed, if u_{n+1}^l lies in the α-th layer in $u_n + (Z^+)^d$, i.e., if

$$\sum_{i=1}^{d} (u_{n+1}^l(i) - u_n(i)) = \alpha - 1,$$

then, by (2.1)

$$\left| \left\{ v \in u_n + (Z^+)^d : \sum_{i=1}^{d} (v(i) - u_n(i)) < \alpha - 1 \right\} \right| < l_n(p),$$

i.e.,

$$\sum_{k=0}^{\alpha-2}\binom{k+d-1}{d-1} < l_n(p). \tag{2.6}$$

Since there exist $C'(d)$ and $C''(d)$ such that

$$\sum_{k=0}^{\alpha-2}\binom{k+d-1}{d-1} \geqslant C'(d)\sum_{k=0}^{\alpha-2}k^{d-1} \geqslant C''(d)\alpha^d,$$

by (2.6) we have

$$|\pi_n| = \alpha \leqslant C(d) l_n^{1/d}(p),$$

where $C(d) = C''(d)^{-1/d}$. Then, since $|\pi_n| \leqslant C(d) l_n^{1/d}(p)$, by (2.2) and (2.5) with $C_1 = C(d)^{-1}$ we have

$$M(p) \geqslant \limsup_{n\to\infty} \frac{n}{\sum_{m=1}^n |\pi_m|} \geqslant \limsup_{n\to\infty} C_1 \frac{n}{\sum_{m=1}^n l_m^{1/d}(p)} \geqslant C_1 p^{1/d}.$$

We now turn to an upper bound for $N(p)$. For this purpose, we need a systematic counting of lattice animals of interest which is Lemma 1 of Cox et al. (1993).

Lemma 1 (Cox, Gandolfi, Griffin and Kesten, 1993). *Let η be a lattice animal of size n and let $1 \leqslant l \leqslant n$ be given. Then, there exists a sequence $\{u_0, \ldots, u_r\}$ in Z^d, of $r+1 \leqslant [(2n-2)/l] + 1$ points, with* $\sup_{1\leqslant k\leqslant d} |u_{i+1}(k) - u_i(k)| \leqslant 1$, $0 \leqslant i < r$, *such that*

$$\eta \subset \bigcup_{i=0}^r B(lu_i, 2l).$$

Moreover, if η contains the origin 0, *then we may in addition choose* $u_0 = 0$.

Let C_2 be a strictly positive finite constant which will be explicitly chosen below. Suppose that there is a lattice animal η of size n containing the origin such that

$$\sum_{v\in\eta} X_v(p) \geqslant C_2 p^{1/d} n.$$

Then, by Lemma 1 with $l = \lceil p^{-1/d} \rceil$, for large n, say, $n \geqslant n_1$, there exists a sequence $\{u_0, \ldots, u_r\}$ in Z^d, of $r+1 \leqslant [(2n-2)/l]+1$ points, with $\sup_{1\leqslant k\leqslant d} |u_{i+1}(k) - u_i(k)| \leqslant 1$, $0 \leqslant i < r$, such that

$$\eta \subset \bigcup_{i=0}^r B(lu_i, 2l).$$

Moreover, since η contains the origin, we can choose $u_0 = 0$. Since η is contained in $\bigcup_{i=0}^r B(lu_i, 2l)$,

$$\sum_{v\in\cup_{i=0}^r B(lu_i,2l)} X_v(p) \geqslant \sum_{v\in\eta} X_v(p) \geqslant C_2 p^{1/d} n.$$

So, for $n \geqslant n_1$

$$
\begin{aligned}
P&\left(\exists \eta \in E(n), \sum_{v\in\eta} X_v(p) \geqslant C_2 p^{1/d} n\right)\\
&\leqslant \sum_{\{u_0,\dots,u_r\}} P\left(\sum_{v\in\bigcup_{i=0}^r B(lu_i,2l)} X_v(p) \geqslant C_2 p^{1/d} n\right)\\
&\leqslant \sum_{\{u_0,\dots,u_r\}} \mathrm{e}^{-C_2 p^{1/d} n} E\left(\prod_{v\in\bigcup_{i=0}^r B(lu_i,2l)} \mathrm{e}^{X_v(p)}\right).
\end{aligned} \tag{2.7}
$$

Note that, since $u_0 = 0$, there are at most $(3d)^r \leqslant (3d)^{(2n-2)/l} \leqslant (3d)^{2np^{1/d}}$ choices for $\{u_0,\dots,u_r\}$. Also note that $\bigcup_{i=0}^r B(lu_i,2l)$ contains, for $n \geqslant n_1$, at most $(r+1)(4l+1)^d \leqslant ([(2n-2)/l]+1)(4l+1)^d \leqslant 3np^{1/d}(9p^{-1/d})^d$ vertices. So, by (2.7) with $C_2 = 2\log(3d) + 6\cdot 9^d$

$$
\begin{aligned}
&\sum_{n=n_1}^{\infty} P(\exists \eta \in E(n), \sum_{v\in\eta} X_v(p) \geqslant C_2 p^{1/d} n)\\
&\leqslant \sum_{n=n_1}^{\infty} \sum_{\{u_0,\dots,u_r\}} \mathrm{e}^{-C_2 p^{1/d} n} E\left(\prod_{v\in\bigcup_{i=0}^r B(lu_i,2l)} \mathrm{e}^{X_v(p)}\right)\\
&\leqslant \sum_{n=n_1}^{\infty} \sum_{\{u_0,\dots,u_r\}} \mathrm{e}^{-C_2 p^{1/d} n} (1 - p + p\mathrm{e})^{3np^{1/d}(9p^{-1/d})^d}\\
&\leqslant \sum_{n=n_1}^{\infty} (3d)^{2np^{1/d}} \mathrm{e}^{-C_2 p^{1/d} n} (1 - p + p\mathrm{e})^{3np^{1/d}(9p^{-1/d})^d}\\
&\leqslant \sum_{n=n_1}^{\infty} (3d)^{2np^{1/d}} \mathrm{e}^{-C_2 p^{1/d} n} \mathrm{e}^{p(\mathrm{e}-1)3np^{1/d}(9p^{-1/d})^d}\\
&= \sum_{n=n_1}^{\infty} \exp(-[-2\log(3d) + C_2 - (\mathrm{e}-1)3\cdot 9^d] p^{1/d} n)\\
&< \infty.
\end{aligned}
$$

Now, by the Borel–Cantelli lemma $N(p) \leqslant C_2 p^{1/d}$ follows from (1.5) and (1.6).

3. Power laws of $M(p)$ and $N(p)$ as $p \uparrow p_c$

In this section, we prove Theorem 3. This again consists of two parts: First, we note that $M(p_c) = 1$. This means that there always exists a long (random) path π such that

$$\sum_{v\in\pi} X_v(p_c) \sim |\pi|.$$

After a coupling, we evaluate the values $X_v(p)$ along this long path and we get a lower bound for $M(p)$. If $d=2$, with the help of some results from first-passage percolation we can get a better lower bound for $M(p)$. Second, we construct a handy tool (Lemma 2) how to construct an upper bound for $N(p)$. In order to use this tool, we count systematically the number of lattice animals of interest and estimate certain probabilities of interest. We then, using the tool, construct an upper bound for $N(p)$.

Let's start with lower bounds for $M(p)$. Let $\{U_v : v \in Z^d\}$ be i.i.d. random variables whose common distribution is the uniform distribution on the unit interval $[0,1]$. Define $\{X_v(p) : v \in Z^d\}$, i.i.d. Bernoulli random variables with a success parameter p, by

$$X_v(p) = I(U_v \leqslant p).$$

Let $\pi_n(p_c)$ be an optimal selfavoiding (random) path of length n starting at the origin for which $\sum_{v\in\pi_n(p_c)} X_v(p_c)$ achieves $M_n(p_c)$. Then, by (1.5) and (1.6) we have for $p<p_c$

$$\begin{aligned} M(p) &\geqslant \limsup_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} X_v(p)}{n} \\ &= \limsup_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p)}{n}. \end{aligned} \tag{3.1}$$

Since $M(p_c)=1$, by (1.5) and (1.6)

$$\lim_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p_c)}{n} = 1. \tag{3.2}$$

So, it makes sense to write (3.1) as

$$\begin{aligned} M(p) &\geqslant \limsup_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p)}{n} \\ &= \limsup_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p_c)}{n} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p)}{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p_c)}. \end{aligned} \tag{3.3}$$

Moreover, by (3.2) and by the strong law of large numbers

$$\lim_{n\to\infty} \frac{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p)}{\sum_{v\in\pi_n(p_c)} I(U_v \leqslant p_c)} = \frac{p}{p_c}. \tag{3.4}$$

Therefore, by (3.2)–(3.4) we have

$$M(p) \geqslant \frac{p}{p_c}. \tag{3.5}$$

If $d=2$, by a well-known result from first-passage percolation on the square lattice Z^2 we get a better lower bound for $M(p)$. Let $\{V_v(p) : v \in Z^d\}$ be defined by

$$V_v(p) = 1 - X_v(p). \tag{3.6}$$

In first-passage percolation, the main interest is the time

$$T_n(p) = \min\left\{ \sum_{v\in\pi} V_v(p) : \pi \text{ a path from } (0,0) \text{ to } (n,0) \right\} \tag{3.7}$$

to travel from $(0,0)$ to $(n,0)$. The fundamental theorem of first-passage percolation is that there exists a finite constant $\mu(p)$ such that

$$\lim_{n\to\infty} \frac{T_n(p)}{n} = \mu(p) \text{ a.s. and in } L^1 \tag{3.8}$$

(see Durrett, 1988, Chapter 8 for more details). $\mu(p)$ is called the *time constant*.

In the case $d=2$, Chayes et al. (1986) study the relation between the time constant $\mu(p)$ and the correlation length $\xi(p)$.

Theorem 4 (Chayes, Chayes and Durrett, 1986). *In* Z^2,

$$\mu(p) \leqslant \frac{C_4}{\xi(p)}. \tag{3.9}$$

Proof. See Theorem 3.2, (2.6), and (2.7) of Chayes et al. (1986). □

Now, let $\tilde{\pi}_n(p)$ be an optimal path from $(0,0)$ to $(n,0)$ for which

$$\sum_{v\in\tilde{\pi}_n(p)} V_v(p) = \min\left\{\sum_{v\in\pi} V_v(p) : \pi \text{ a path from } (0,0) \text{ to } (n,0)\right\}$$

and let $\pi_n^*(p)$ be the initial piece of the path $\tilde{\pi}_n(p)$ of n vertices. Then, by (3.6)–(3.9)

$$\begin{aligned} M(p) &\geqslant \limsup_{n\to\infty} \frac{\sum_{v\in\pi_n^*(p)} X_v(p)}{n} \\ &= \limsup_{n\to\infty} \frac{n - \sum_{v\in\pi_n^*(p)} V_v(p)}{n} \\ &\geqslant \lim_{n\to\infty} \frac{n - \sum_{v\in\tilde{\pi}_n(p)} V_v(p)}{n} \\ &= 1 - \mu(p) \\ &\geqslant 1 - C_4/\xi(p). \end{aligned}$$

Now, let us construct an upper bound for $N(p)$. To do this, we need a tool how to construct an upper bound of $N(p)$. The next Lemma provides us such a tool.

Lemma 2. *Let* $\{X_v(p) : v\in Z^d\}$ *be i.i.d. Bernoulli random variables with a success parameter* p. *Assume that there exist* c, C_5, *and* n_2 *such that for* $n\geqslant n_2$

$$P\left(\exists\eta \in \bigcup_{m=n}^{\infty} E(m), \sum_{v\in\eta}(1 - X_v(p)) \leqslant cn\right) \leqslant e^{-C_5 n}. \tag{3.10}$$

Then,

$$N(p) \leqslant 1 - c. \tag{3.11}$$

Proof. If there is a lattice animal η of size n containing the origin such that $\sum_{v\in\eta} X_v(p) \geqslant (1-c)n$, then

$$\sum_{v\in\eta}(1 - X_v(p)) \leqslant cn.$$

So, for $n \geqslant n_2$, by (3.10)

$$P\left(\frac{N_n(p)}{n} \geqslant 1 - c\right) \leqslant P\left(\exists\eta \in \bigcup_{m=n}^{\infty} E(m), \sum_{v\in\eta}(1 - X_v(p)) \leqslant cn\right) \leqslant \mathrm{e}^{-C_5 n}.$$

Therefore, by the Borel–Cantelli lemma

$$\frac{N_n(p)}{n} \leqslant 1 - c$$

for large n with probability 1 and hence (3.11) follows from (1.5) and (1.6). □

Fix p_0 such that $0 < p_0 < p_c$. Let L be a strictly positive finite constant which will be explicitly chosen below. For a lattice animal η of size $|\eta| \geqslant n$ containing the origin, we construct a "covering" lattice animal

$$\bar{\eta} := \{v \in Z^d : B(Lv, L) \cap \eta \neq \emptyset\}$$

of size $|\bar{\eta}| \geqslant n/3^d(2L+1)^d$. Note that, since η contains the origin, so does $\bar{\eta}$. Construct a spanning tree τ for $\bar{\eta}$ with root 0. Then, τ has $|\bar{\eta}|$ vertices and hence $|\bar{\eta}| - 1$ edges. Construct a path $\pi = (v_1, \dots, v_k)$ in τ (which may not be selfavoiding) starting at the origin $v_1 = 0$ which contains all vertices in τ with (at most twice as many edges as τ and hence) at most $2|\bar{\eta}| - 1$ vertices (see Durrett et al., 1991), Section 2 for the explicit construction of such a π). Since π visits all $|\bar{\eta}|$ vertices in $\bar{\eta}$, we can construct a sequence $u_1, \dots, u_l$, $l \geqslant |\bar{\eta}|/7^d$, such that

$$B(Lu_i, L) \cap \eta \neq \emptyset, \quad 1 \leqslant i \leqslant l, \tag{3.12}$$

and

$$B(Lu_i, 3L) \cap B(Lu_j, 3L) = \emptyset, \quad i \neq j, \tag{3.13}$$

in the following way: Let $u_0 = 0$. Once u_n has been chosen, we choose u_{n+1} the first vertex v in π after u_n such that

$$B(Lv, 3L) \cap B(Lu_i, 3L) = \emptyset, \quad 1 \leqslant i \leqslant n.$$

Since π visits all $|\bar{\eta}|$ vertices in $\bar{\eta}$ and since $B(Lu_i, 3L)$ contains exactly 7^d vertices of the form Lv, $v \in Z^d$, indeed we can construct a sequence $u_1, \dots, u_l$, $l \geqslant |\bar{\eta}|/7^d$, which satisfies (3.12) and (3.13).

Suppose that there exists a lattice animal η of size $|\eta| \geqslant n$ containing the origin such that

$$\sum_{v\in\eta}(1 - X_v(p)) \leqslant cn.$$

Then, as we described above, we can construct a sequence $u_1,\dots,u_l$, $l \geqslant |\bar{\eta}|/7^d$, which satisfies (3.12) and (3.13). For large $n \geqslant n_3 = n_3(L)$, $l \geqslant 2$, therefore $\eta \cap B(Lu_i, L) \neq \emptyset$ and $\eta \not\subset B(Lu_i, 3L)$, and hence for $1 \leqslant i \leqslant l$ there exists a selfavoiding path π_i from $\partial B(Lu_i, L)$ to $\partial B(Lu_i, 3L)$ in $\eta \cap B(Lu_i, 3L)$. So, since $\pi_i \subset \eta$ and $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$,

$$\sum_{i=1}^{l} t(\partial B(Lu_i, L), \partial B(Lu_i, 3L)) \leqslant \sum_{i=1}^{l} \sum_{v \in \pi_i} (1 - X_v(p)) \leqslant \sum_{v \in \eta} (1 - X_v(p)) \leqslant cn,$$

where for two disjoint subsets $\mathscr{A}$ and $\mathscr{B}$ of Z^d

$$t(\mathscr{A}, \mathscr{B}) = \inf \left\{ \sum_{v \in \pi} (1 - X_v(p)) : \pi \text{ a selfavoiding path from } \mathscr{A} \text{ to } \mathscr{B} \right\}.$$

Note here that $1 - X_v(p)$ now takes over the role of $X_v(p)$ in the usual first-passage percolation set up. Since $\{t(\partial B(Lu_i, L), \partial B(Lu_i, 3L)\}$ are i.i.d. random variables whose common distribution is the same as that of $t(\partial B(L), \partial B(3L)$, for $n \geqslant n_3$

$$\begin{aligned}
& P\left(\exists \eta \in \bigcup_{m=n}^{\infty} E(m), \sum_{v \in \eta} (1 - X_v(p)) \leqslant cn \right) \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} \; \sum_{\{u_1,\dots,u_l\}} P\left(\sum_{i=1}^{l} t(\partial B(Lu_i, L), \partial B(Lu_i, 3L)) \leqslant cn \right) \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} \; \sum_{\{u_1,\dots,u_l\}} \mathrm{e}^{\lambda cn} E\left(\mathrm{e}^{-\lambda \sum_{i=1}^{l} t(\partial B(Lu_i, L), \partial B(Lu_i, 3L))} \right) \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} \; \sum_{\{u_1,\dots,u_l\}} \mathrm{e}^{\lambda cn} [E(\mathrm{e}^{-\lambda t(\partial B(L), \partial B(3L))})]^l. \qquad (3.14)
\end{aligned}$$

Note that for a lattice animal $\bar{\eta}$ of size $|\bar{\eta}|$ containing the origin there are at most $(2d)^{2|\bar{\eta}|-2}$ paths π of length $2|\bar{\eta}| - 1$ staring at the origin which covers $\bar{\eta}$ and hence there are at most $(2d)^{2|\bar{\eta}|-2}$ choices for $\{u_1,\dots,u_l\}$. So, by (3.14)

$$\begin{aligned}
& P\left(\exists \eta \in \bigcup_{m=n}^{\infty} E(m), \sum_{v \in \eta} (1 - X_v(p)) \leqslant cn \right) \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} \; \sum_{\{u_1,\dots,u_l\}} \mathrm{e}^{\lambda cn} [E(\mathrm{e}^{-\lambda t(\partial B(L), \partial B(3L))})]^l \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} (2d)^{2|\bar{\eta}|-2} \mathrm{e}^{\lambda cn} [E(\mathrm{e}^{-\lambda t(\partial B(L), \partial B(3L))})]^l \\
& \leqslant \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} (4d^2)^{|\bar{\eta}|} \mathrm{e}^{\lambda c 3^d (2L+1)^d |\bar{\eta}|} E(\mathrm{e}^{-\lambda t(\partial B(L), \partial B(3L))})]^{|\bar{\eta}|/7^d} \\
& = \sum_{|\bar{\eta}| \geqslant n/3^d(2L+1)^d} [4d^2 \mathrm{e}^{\lambda c 3^d (2L+1)^d} (E(\mathrm{e}^{-\lambda t(\partial B(L), \partial B(3L))}))^{1/7^d}]^{|\bar{\eta}|}. \qquad (3.15)
\end{aligned}$$

If $\partial B(L) \leftrightarrow \partial B(3L)$, then there exists some vertex $v \in \partial B(L)$ such that $v \leftrightarrow \partial B(3L)$ and hence $v \leftrightarrow \partial B(v, 2L)$. So, by (1.2) and (1.3) we can choose C_6, independent of $p_0 < p < p_c$, large so that

$$L = C_6 \xi(p) \log \xi(p)$$

satisfies for $p_0 < p < p_c$

$$\begin{aligned} &4d^2(P(t(\partial B(L), \partial B(3L)) = 0))^{1/7^d} \\ &\quad = 4d^2(P_p(\partial B(L) \leftrightarrow \partial B(3L)))^{1/7^d} \\ &\quad \leqslant 4d^2((2L+1)^d P_p(0 \leftrightarrow \partial B(2L)))^{1/7^d} \\ &\quad \leqslant \tfrac{1}{4}. \end{aligned} \tag{3.16}$$

Since $t(\partial B(L), \partial B(3L)) \geqslant 1$ if $t(\partial B(L), \partial B(3L)) \neq 0$, we can choose $\lambda = C_7$, independent of $p_0 < p < p_c$, large so that for $p_0 < p < p_c$

$$\begin{aligned} &4d^2(E(e^{-\lambda t(\partial B(L), \partial B(3L))}))^{1/7^d} \\ &\quad \leqslant 4d^2(P(t(\partial B(L), \partial B(3L)) = 0) + e^{-\lambda} P(t(\partial B(L), \partial B(3L)) \neq 0))^{1/7^d} \\ &\quad \leqslant \tfrac{1}{2}. \end{aligned} \tag{3.17}$$

Finally, we can choose C_3, independent of $p_0 < p < p_c$, small so that for $p_0 < p < p_c$

$$c = C_3(\xi(p) \log \xi(p))^{-d}$$

satisfies for $p_0 < p < p_c$

$$4d^2 e^{\lambda c 3^d (2L+1)^d} (E(e^{-\lambda t(\partial B(L), \partial B(3L))}))^{1/7^d} \leqslant \tfrac{3}{4}. \tag{3.18}$$

Therefore, by (3.14)–(3.18) with $c = C_3(\xi(p) \log \xi(p))^{-d}$ we have for $\geqslant n_3$

$$P\left(\exists \eta \in \bigcup_{m=n}^{\infty} E(m), \sum_{v \in \eta} (1 - X_v(p)) \leqslant cn\right) \leqslant 4 \exp\left[-\frac{\log 4/3}{3^d(2L+1)^d} n\right].$$

Therefore, by Lemma 2 we have

$$N(p) \leqslant 1 - \frac{C_3}{(\xi(p) \log \xi(p))^d}.$$

Acknowledgements

This paper is a part of the author's Ph.D. thesis under Professor Harry Kesten. He would like to thank Professor Kesten for his insightful suggestions and encouragement.

References

Chayes, J.T., Chayes, L., Durrett, R., 1986. Critical behavior of the two-dimensional first passage time. J. Statist. Phys. 45, 933–951.

Cox, J.T., Gandolfi, A., Griffin, Ph.S., Kesten, S., 1993. Greedy lattice animals I: upper bounds. Ann. Appl. Probab. 3, 1151–1169.

Durrett, R., 1988. Lecture Notes on Particle Systems and Percolation. Wadsworth, Berlin.

Durrett, R., Kesten, H., Waymire, E., 1991. On weighted heights of random trees. J. Theoret. Probab. 4, 223–237.

Fontes, L., Newman, C.M., 1993. First passage percolation for random colorings of Z^d. Ann. Appl. Probab. 3, 746–762.

Gandolfi, A., Kesten, H., 1994. Greedy lattice animals II: linear growth. Ann. Appl. Probab. 4, 76–107.

Grimmett, G., 1989. Percolation. Springer, Berlin.

Lee, S., 1993. An inequality for greedy lattice animals. Ann. Appl. Probab. 3, 1170–1188.

Lee, S., 1994. A note on greedy lattice animals. Ph.D. Thesis, Cornell University.

Lee, S., 1997. The continuity of M and N in greedy lattice animals. J. Theoret. Probab., 10, 87–100.

Sznitman, A., 1995. Crossing velocities and random lattice animals. Ann. Probab. 23, 1006–1023.